ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 2, pp: (1-4), Month: October 2024 – March 2025, Available at: www.paperpublications.org

Fractional Differential Problem of Two Matrix Fractional Hyperbolic Functions

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.14000570

Published Date: 28-October-2024

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of two matrix fractional hyperbolic functions. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional hyperbolic functions.**

I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in physics, mechanics, viscoelasticity, economics, mathematical biology, electrical engineering, control theory, and other fields [1-14]. However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [15-19]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional derivative, we obtain arbitrary order fractional derivative of the following two matrix fractional hyperbolic functions:

$$[cosh_{\alpha}(rAx^{\alpha})]^{\otimes_{\alpha} m}\ , \tag{1}$$

and

$$[sinh_{\alpha}(rAx^{\alpha})]^{\otimes_{\alpha} m}\ , \tag{2}$$

where $0 < \alpha \leq 1$, m is a positive integer, r is a real number, and A is a matrix. A new multiplication of fractional analytic functions plays an important role in this article. Moreover, our results are generalizations of the results in classical calculus.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper.

Definition 2.1 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{3}$$

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 2, pp: (1-4), Month: October 2024 – March 2025, Available at: www.paperpublications.org

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([21])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x - x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x - x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0 \quad . \tag{5}$$

Definition 2.3 ([22])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([23])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}. \tag{7}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x - x_0)^{n\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{9}$$

Definition 2.5: If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \tag{10}$$

In addition, the matrix α-fractional hyperbolic cosine and hyperbolic sine function are defined as follows:

$$cosh_\alpha(Ax^\alpha) = \frac{1}{2}[E_\alpha(Ax^\alpha) + E_\alpha(-Ax^\alpha)] = \sum_{n=0}^{\infty} A^{2n} \frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2n}, \tag{11}$$

and

$$sinh_\alpha(Ax^\alpha) = \frac{1}{2}[E_\alpha(Ax^\alpha) - E_\alpha(-Ax^\alpha)] = \sum_{n=0}^{\infty} A^{2n+1} \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2n+1)}, \tag{12}$$

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 2, pp: (1-4), Month: October 2024 – March 2025, Available at: www.paperpublications.org

Theorem 2.6 (fractional binomial theorem)**:** *If* $0<\alpha\leq 1$, m *is a positive integer and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are two* α*-fractional analytic functions. Then*

$$[f_\alpha(x^\alpha)+g_\alpha(x^\alpha)]^{\otimes_\alpha m}=\sum_{k=0}^{n}\binom{m}{k}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (m-k)}\otimes_\alpha\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\ , \tag{13}$$

where $\binom{m}{k}=\frac{m!}{k!(m-k)!}$.

III. MAIN RESULTS

In this section, we find arbitrary order fractional derivative of two matrix fractional hyperbolic functions.

Theorem 3.1: *Let* $0<\alpha\leq 1$, m,p *be positive integers,* r *be a real number, and* A *be a matrix. Then*

$$\left({}_0D_x^\alpha\right)^p\left[[cosh_\alpha(rAx^\alpha)]^{\otimes_\alpha m}\right]=\frac{1}{2^m}r^pA^p\sum_{k=0}^{m}\binom{m}{k}(m-2k)^pE_\alpha((m-2k)rAx^\alpha). \tag{14}$$

Proof $\left({}_0D_x^\alpha\right)^p\left[[cosh_\alpha(rAx^\alpha)]^{\otimes_\alpha m}\right]$

$$=\left({}_0D_x^\alpha\right)^p\left[\left[\frac{1}{2}[E_\alpha(rAx^\alpha)+E_\alpha(-rAx^\alpha)]\right]^{\otimes_\alpha m}\right]$$

$$=\frac{1}{2^m}\left({}_0D_x^\alpha\right)^p\left[\sum_{k=0}^{m}\binom{m}{k}[E_\alpha(rAx^\alpha)]^{\otimes_\alpha (m-k)}\otimes_\alpha[E_\alpha(-rAx^\alpha)]^{\otimes_\alpha k}\right]$$ (by fractional binomial theorem)

$$=\frac{1}{2^m}\left({}_0D_x^\alpha\right)^p\left[\sum_{k=0}^{m}\binom{m}{k}[E_\alpha((m-2k)rAx^\alpha)]\right]$$

$$=\frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\left({}_0D_x^\alpha\right)^p[E_\alpha((m-2k)rAx^\alpha)]$$

$$=\frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}(m-2k)^pr^pA^pE_\alpha((m-2k)rAx^\alpha)$$

$$=\frac{1}{2^m}r^pA^p\sum_{k=0}^{m}\binom{m}{k}(m-2k)^pE_\alpha((m-2k)rAx^\alpha).$$ q.e.d.

Theorem 3.2: *Suppose that* $0<\alpha\leq 1$, m,p *are positive integers,* r *is a real number, and* A *is a matrix. Then*

$$\left({}_0D_x^\alpha\right)^p\left[[sinh_\alpha(rAx^\alpha)]^{\otimes_\alpha m}\right]=\frac{1}{2^m}r^pA^p\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-2k)^pE_\alpha((m-2k)rAx^\alpha). \tag{15}$$

Proof $\left({}_0D_x^\alpha\right)^p\left[[sinh_\alpha(rAx^\alpha)]^{\otimes_\alpha m}\right]$

$$=\left({}_0D_x^\alpha\right)^p\left[\left[\frac{1}{2}[E_\alpha(rAx^\alpha)-E_\alpha(-rAx^\alpha)]\right]^{\otimes_\alpha m}\right]$$

$$=\frac{1}{2^m}\left({}_0D_x^\alpha\right)^p\left[\sum_{k=0}^{m}\binom{m}{k}[E_\alpha(rAx^\alpha)]^{\otimes_\alpha (m-k)}\otimes_\alpha[-E_\alpha(-rAx^\alpha)]^{\otimes_\alpha k}\right]$$ (by fractional binomial theorem)

$$=\frac{1}{2^m}\left({}_0D_x^\alpha\right)^p\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^k[E_\alpha((m-2k)rAx^\alpha)]\right]$$

$$=\frac{1}{2^m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left({}_0D_x^\alpha\right)^p[E_\alpha((m-2k)rAx^\alpha)]$$

$$=\frac{1}{2^m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-2k)^pr^pA^pE_\alpha((m-2k)rAx^\alpha)$$

$$=\frac{1}{2^m}r^pA^p\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-2k)^pE_\alpha((m-2k)rAx^\alpha).$$ q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of two matrix fractional hyperbolic functions. In fact, our results are generalizations of classical calculus results. In the future, we will continue to use Jumarie type of R-L fractional derivative and the new multiplication of fractional analytic functions to solve the problems in engineering mathematics and fractional differential equations.

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 11, Issue 2, pp: (1-4), Month: October 2024 – March 2025, Available at: www.paperpublications.org

REFERENCES

[1] M. Stiassnie, On the application of fractional calculus for the formulation of viscoelastic models, Applied Mathematical Modelling, Vol. 3, pp. 300-302, 1979.

[2] R. Magin, Fractional calculus in bioengineering, part 1, Critical Reviews in Biomedical Engineering, Vol. 32, No,1. pp.1-104, 2004.

[3] R. Hilfer, Ed., Applications of fractional calculus in physics, World Scientific Publishing, Singapore, 2000.

[4] J. A. T. Machado, Analysis and design of fractional-order digital control systems, Systems Analysis Modelling Simulation, vol. 27, no. 2-3, pp. 107-122, 1997.

[5] H. A. Fallahgoul, S. M. Focardi and F. J. Fabozzi, Fractional calculus and fractional processes with applications to financial economics, theory and application, Elsevier Science and Technology, 2016.

[6] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[7] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[8] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics, vol.23, pp.397-404, 2002.

[9] M. F. Silva, J. A. T. Machado, and I. S. Jesus, Modelling and simulation of walking robots with 3 dof legs, in Proceedings of the 25th IASTED International Conference on Modelling, Identification and Control (MIC '06), pp. 271-276, Lanzarote, Spain, 2006.

[10] F. Duarte and J. A. T. Machado, Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators, Nonlinear Dynamics, vol. 29, no. 1-4, pp. 315-342, 2002.

[11] M. Teodor, Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[12] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[13] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[14] F. Mainardi, Fractional Calculus: Theory and Applications, Mathematics, vol. 6, no. 9, 145, 2018.

[15] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[16] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., 1974.

[17] S. Das, Functional Fractional Calculus, 2nd ed. Springer-Verlag, 2011.

[18] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[19] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, USA, 1993.

[20] C. -H. Yu, Application of differentiation under fractional integral sign, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 2, pp. 40-46, 2022.

[21] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[22] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[23] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.